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ON OPTIMAL SUBSPACES FOR KOLMOGOROV WIDTHS OF CLASSES OF 2π-PERIODIC ANALYTIC FUNCTIONS

Summary

The following problem arises: to find optimal subspaces for Kolmogorov widths of classes of convolutions with generated kernels, which may increase the oscillations.

Keywords and phrases: Kolmogorov width, optimal subspace, Poisson kernel, kernel of analytic functions, CVD-kernels

Let X be a linear normed space and let $\mathfrak{N}$ a centrally symmetric set from X. The quantity

$$d_m(\mathfrak{N}, X) = \inf_{F_m \subset X} \sup_{f \in \mathfrak{N}} \inf_{g \in F_m} \|f - g\|_X, \ m \in \mathbb{N}, \tag{1}$$

is called *the m-dimensional Kolmogorov width* where the external inf is taken over all possible m-dimensional linear subspaces F_m from X [1].

The subspace $F_m^* \subset X$ for which the equality

$$d_m(\mathfrak{N}, X) = E(\mathfrak{N}, F_m^*)_X = \sup_{f \in \mathfrak{N}} \inf_{g \in F_m^*} \|f - g\|_X$$

is satisfied, is called *the extremal subspace* for width $d_m(\mathfrak{N}, X)$.

We consider the space $L = L_1$ functions φ of 2π-periodic summable on $[0, 2\pi)$ with the norm

$$\|\varphi\|_L = \|\varphi\|_1 = \int_0^{2\pi} |\varphi(t)| dt,$$

Let L_∞ be the space of measurable and essentially bounded 2π-periodic functions φ with the norm

$$\|\varphi\|_\infty = \operatorname*{ess\,sup}_{t \in \mathbb{R}} |\varphi(t)|$$

and C the space of continuous 2π-periodic functions φ with the norm

$$\|\varphi\|_C = \max_{t\in\mathbb{R}} |\varphi(t)|.$$

Let $C^{\psi}_{\beta,p}$ denote the class of 2π-periodic functions f, representable as convolutions of the form

$$f(x) = A + \big(\Psi_\beta * \varphi\big)(x) = A + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \Psi_\beta(x-t)\varphi(t)dt, \quad A \in \mathbb{R}, \tag{2}$$

where $\varphi \perp 1, \quad \|\varphi\|_p \le 1, \ p = 1, \infty$ and Ψ_β is a fixed kernel of the form

$$\Psi_\beta(t) = \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta\pi}{2}\right), \ \psi(k) > 0, \ \sum_{k=1}^{\infty} \psi(k) < \infty, \quad \beta \in \mathbb{R}.$$

In the case, when $\psi(k) = q^k$, $q \in (0,1)$, the classes $C^{\psi}_{\beta,p}$ are classes of convolutions with the Poisson kernels $P_{q,\beta}(t)$ of the form

$$P_{q,\beta}(t) = \sum_{k=1}^{\infty} q^k \cos\left(kt - \frac{\beta\pi}{2}\right), \ q \in (0,1), \quad \beta \in \mathbb{R}; \tag{3}$$

we denote them by $C^{q}_{\beta,p}$. These classes consist (see, e.g. [2, Ch. 3, §8]) of functions that admit a regular extension to the strip $|\operatorname{Im} z| < \ln 1/q$ in the complex plane.

If $\psi(k) = \dfrac{1}{\operatorname{ch} kh}$, $h > 0$, then the classes $C^{\psi}_{\beta,p}$ are denoted by $C^{h}_{\beta,p}$ and are classes of convolutions with kernels of analytic functions

$$H_{h,\beta}(t) = \sum_{k=1}^{\infty} \frac{1}{\operatorname{ch} kh} \cos\left(kt - \frac{\beta\pi}{2}\right), \ h > 0, \ \beta \in \mathbb{R}. \tag{4}$$

Functions from the classes $C^{h}_{\beta,p}$ admit a regular extension to the strip $|\operatorname{Im} z| < h$ in the complex plane (see, e.g. [2, Ch. 3, §8]).

In works [3]-[11] the exact values of widths $d_{2n-1}(C^{q}_{\beta,\infty}, C)$, $d_{2n}(C^{q}_{\beta,\infty}, C)$, $d_{2n-1}(C^{q}_{\beta,1}, L)$, $d_{2n-1}(C^{h}_{\beta,\infty}, C)$, $d_{2n}(C^{h}_{\beta,\infty}, C)$, $d_{2n-1}(C^{h}_{\beta,1}, L)$ were found for all numbers n, beginning with some numbers n_q and n_h, respectively. Herewith, it was shown that in the case $\psi(k) = q^k$, $q \in (0,1)$, or $\psi(k) = \dfrac{1}{\operatorname{ch} kh}$, $h > 0$, the equalities take place:

$$\begin{aligned} &d_{2n-1}(C^{\psi}_{\beta,\infty}, C) = d_{2n}(C^{\psi}_{\beta,\infty}, C) = d_{2n-1}(C^{\psi}_{\beta,1}, L) = \\ &= E(C^{\psi}_{\beta,\infty}, \mathcal{T}_{2n-1})_C = E(C^{\psi}_{\beta,1}, \mathcal{T}_{2n-1})_L = \|\Psi_\beta * \operatorname{sign}\sin(\cdot)\|_C, \end{aligned} \tag{5}$$

where $\mathcal{T}_{2n-1}$ is the subspace of trigonometric polynomials T_{n-1} of order $n-1$:

$$T_{n-1}(t) = a_0 + \sum_{k=1}^{n-1} a_k \cos kt + b_k \sin kt, \quad a_k, b_k \in \mathbb{R}.$$

Equalities (5) imply, that subspaces $\mathcal{T}_{2n-1}$ of trigonometric polynomials T_{n-1} of order $n-1$ are optimal for odd widths $d_{2n-1}(C^{\psi}_{\beta,\infty}, C)$ and $d_{2n-1}(C^{\psi}_{\beta,1}, L)$.

At the same time, as it follows from [12, pp. 180, 183] in the case, where kernels Ψ_β of convolutions of the form (2) are CVD-kernels (Cyclic Variation Diminishing Kernels), i.e.

$$\nu(\Psi_\beta * \varphi) \le \nu(\varphi) \quad \forall \varphi \in C,$$

where $\nu(g)$ is the number of changes of the sign of a function $g \in C$ on $[0, 2\pi)$, the equalities (5) take place for all positive integers n and subspaces

$$S_{2n}(\psi, \beta) = \operatorname{span} \Big\{ \Psi_\beta\Big(\cdot - \frac{k\pi}{n} \Big) \Big\}_{k=1}^{2n} \tag{6}$$

are optimal subspaces for widths $d_{2n}(C^\psi_{\beta,\infty}, C)$.

The kernels $H_{h,0}(t) = \sum\limits_{k=1}^{\infty} \frac{1}{\operatorname{ch} kh} \cos kt$ are CVD-kernels (see, e.g. [12, p. 128]), so all said above is true for the classes $C^h_{0,\infty}$ generated by these kernels . In that time in general situation, i.e. for arbitrary $\beta \in \mathbb{R}$, kernels $P_{q,\beta}(t)$ of the form (3) and kernels $H_{h,\beta}(t)$ of the form (4) may not be CVD-kernels. Then the following problem arises: *to find out (to prove or to disprove): are the spaces $S_{2n}(\psi, \beta)$ of the form (6) optimal for widths $d_{2n}(C^\psi_{\beta,\infty}, C)$ in the case where $\psi(k) = q^k$, $q \in (0, 1)$, or $\psi(k) = \frac{1}{\operatorname{ch} kh}$, $h > 0$, for all $\beta \in \mathbb{R}$ and for all positive integers n, for which the equalities (5) are satisfied.*

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O OPTYMALNYCH PODPRZESTRZENIACH DLA SZEROKOŚCI KOŁMOGOROWA KLAS 2π-OKRESOWYCH FUNKCJI ANALITYCZNYCH

S t r e s z c z e n i e

Stawiamy następujący problem: znaleźć optymalne podprzestrzenie dla szerokości Kołmogorowa klas splotów z wygenerowanymi jądrami, które mogą powiększyć oscylację.

Słowa kluczowe: szerokość Kołmogorowa, optymalna podprzestrzeń, jądro Poissona, jądro funkcji analitycznych, jądro zmniejszające cykliczną wariancję