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Dedicated to the memory of
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PROBLEM ON EXTREMAL DECOMPOSITION OF THE COMPLEX PLANE

Summary

The paper is devoted to one extremal problem in geometric function theory of complex variables associated with estimates of functionals defined on the systems of non-overlapping domains. We consider Dubinin's problem of the maximum of product of inner radii of n non-overlapping domains containing points of the unit circle and the power γ of the inner radius of a domain containing the origin. The problem was formulated in 1994 in the work of Dubinin and then repeated in his monograph in 2014. Currently it is not solved in general. In this paper we generalized it to the case of the more general system of points and obtained a solution of this problem for some concrete values of n and γ.

Keywords and phrases: inner radius of domain, non-overlapping domains, radial system of points, separating transformation, quadratic differential, Green's function

Extremal problems on non-overlapping domains constitute known classical direction of the geometric function theory of a complex variable. A lot of such problems are reduced to determination of the maximum of product of inner radii on the system of non-overlapping domains satisfying a certain conditions. Start point of the theory of extremal problems on non-overlapping domains is the result of Lavrentev [1] who in 1934 solved the problem of product of conformal radii of two mutually non-overlapping simply connected domains.

Theorem 1. *[1] Let a_1 and a_2 be some fixed points of the complex plane $\mathbb{C}$, B_k, $a_k \in B_k$, $k = 1, 2$ be an arbitrary mutually non-overlapping domains of $\overline{\mathbb{C}}$, and* *functions $w = f_k(z)$, $k = 1, 2$, are regular in the circle $\{z : |z| < 1\}$ and univalently map unit circle onto domains B_k, $k = 1, 2$, so that $f_k(0) = a_k$, $k = 1, 2$. Then the following inequality holds*

$$|f_1'(0)| \cdot |f_2'(0)| \leq |a_1 - a_2|^2 .$$

Equality in the inequality is achieved iff

$$B_1 = \left\{ w \in \mathbb{C} : \left| \frac{w - a_1}{w - a_2} \right| < 1 \right\}, \quad B_2 = \left\{ w \in \mathbb{C} : \left| \frac{w - a_1}{w - a_2} \right| > 1 \right\}.$$

Goluzin in [2] generalized this theorem on the case of an arbitrary finite number of mutually disjoint domains and obtained an accurate evaluation for the case of three domains

$$\prod_{k=1}^{3} |f_k'(0)| \leq \frac{64}{81\sqrt{3}} \cdot |a_1 - a_2| \cdot |a_1 - a_3| \cdot |a_2 - a_3|.$$

Further Kuzmina [3] showed that the problem of the evaluation for the case of four domains is reduced to the smallest capacity problems in the certain continuum family and got the exact inequality for $n = 4$

$$\prod_{k=1}^{4} |f_k'(0)| \leq \frac{9}{4^{\frac{8}{3}}} \left(|a_1 - a_2| \cdot |a_1 - a_3| \cdot |a_2 - a_3| \cdot |a_1 - a_4| \cdot |a_2 - a_4| \cdot |a_3 - a_4| \right)^{\frac{2}{3}} .$$

For $n \geq 5$ full solution of the problem is not obtained at this time. Since, the evaluation of the product of conformal radii of mutually non-overlapping domains if $n \geq 5$ without any restriction on the domains B_k and points a_k, $k = 1, ..., 5$ is quite difficult and interesting problem.

In 1955 Kolbina [4] generalized the Lavrentev result adding some degrees α and β to conformal radii and obtained the inequality

$$|f_1'^{\alpha}(0)| \cdot \left| f_2'^{\beta}(0) \right| \leq |a_1 - a_2|^{\alpha + \beta} \cdot A_{\alpha\beta},$$

where

$$A_{\alpha\beta} = \frac{4^{\alpha+\beta} \alpha^{\alpha} \beta^{\beta}}{|\alpha - \beta|^{\alpha+\beta}} \left[\frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{\alpha} + \sqrt{\beta}} \right]^{2\sqrt{\alpha\beta}}, \quad A_{\alpha\alpha} = 1.$$

In 1975 Lebedev [5] considered the more general extremal problem of product of conformal radii.

Problem 1. [5] There are n various fixed points a_k, $k = \overline{1, n}$, $n > 3$, on a plane w. Functions $w = f_k(z)$, $k = \overline{1, n}$, are regular in the circle $|z| < 1$ and univalent map circle $|z| < 1$ onto non-overlapping domains B_k, which contain the corresponding points a_k, $k = \overline{1, n}$, and in such a way, that $f_k(0) = a_k$, $k = \overline{1, n}$. What about maximum of product

$$\prod_{k=1}^{n} |f_k'(0)|^{\gamma_k} \longrightarrow \max, \quad \gamma_k > 0, \, n > 3,$$

relatively to any functions $f_k(z)$, $k=\overline{1,n}$?

However, this problem is generally not solved so far. Further Problem 1 was generalized to more general classes of multiply connected domains replacing conformal radius to the inner radius.

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}}=\mathbb{C}\bigcup\{\infty\}$ be a one point compactification and $\mathbb{R}^+=(0,\infty)$. Let $\chi(t)=\frac{1}{2}(t+t^{-1})$, $t\in\mathbb{R}^+$, be the function of Zhukovsky. Let B be a domain in $\mathbb{C}$, $a\in B$ be a point in B and $r(B,a)$ be an inner radius of the domain $B\subset\overline{\mathbb{C}}$ with respect to the point $a\in B$. Inner radius is a generalization of conformal radius for multiply connected domains.

Inner radius of the domain B is associated with a generalized Green's function $g_B(z,a)$ of the domain B by the relations

$$g_B(z,a)=\ln\frac{1}{|z-a|}+\ln r(B,a)+o(1),\quad z\to a.$$
$$g_B(z,\infty)=\ln|z|+\ln r(B,\infty)+o(1),\quad z\to\infty.$$

Let $n\in\mathbb{N}$. A set of points $A_n:=\left\{a_k\in\mathbb{C}:\ k=\overline{1,n}\right\}$, $n\in\mathbb{N}$, $n\geq 2$ is called n-radial system if $|a_k|\in\mathbb{R}^+$, $k=\overline{1,n}$ and

$$0=\arg a_1<\arg a_2<\ldots<\arg a_n<2\pi.$$

Denote $a_{n+1}:=a_1$, $\alpha_k:=\frac{1}{\pi}\arg\frac{a_{k+1}}{a_k}$, $\alpha_{n+1}:=\alpha_1$, $k=\overline{1,n}$, $\sum\limits_{k=1}^{n}\alpha_k=2$.

For an arbitrary n-radial system of points $A_n=\{a_k\}_{k=1}^n$ and $\gamma\in\mathbb{R}^+\cup\{0\}$ we introduce the "control" functional

$$\mathcal{L}^{(\gamma)}(A_n):=\prod_{k=1}^{n}\left[\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\right]^{1-\frac{1}{2}\gamma\alpha_k^2}\prod_{k=1}^{n}|a_k|^{1+\frac{1}{4}\gamma(\alpha_k+\alpha_{k-1})}.$$

If $\gamma=0$ then

$$\mathcal{L}^{(0)}(A_n):=\prod_{k=1}^{n}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\cdot|a_k|.$$

It is clear that the class of n-radial systems of points for which $\mathcal{L}^{(\gamma)}(A_n)=1$ automatically includes all systems of n distinct points that are located on the unit circle.

Note that to describe the extremal configurations of domains we use notion of quadratic differential (see, for example, [6, 7]). Quadratic differential $G(z)dz^2$ on a Riemann surface is a rule which associates to each local parameter z mapping a parametric neighbourhood $U\subset\mathbb{R}$ into the extended complex plane $\overline{\mathbb{C}}(z:U\to\overline{\mathbb{C}})$, a function $G_z:z(U)\to\overline{\mathbb{C}}$ such that for any local parameters $z_1:U_1\to\overline{\mathbb{C}}$ and $z_2:U_2\to\overline{\mathbb{C}}$ with $U_1\cap U_2$ non-empty, the following holds in this intersection

$$\frac{G_{z_2}(z_2(p))}{G_{z_1}(z_1(p))}=\left(\frac{dz_1(p)}{dz_2(p)}\right)^2,\quad p\in U_1\cap U_2,$$

here $z(U)$ is the image of U in $\overline{\mathbb{C}}$ under z. In other words, a quadratic differential is a non-linear differential of type $(2,0)$ on a Riemann surface. The functions entering into the definition of a quadratic differential are ordinarily assumed to be measurable or even analytic.

Consider an extremal problem which in the case of a unit circle was formulated in 1994 in the paper of Dubinin [8, P.68, no.9.2] in the list of unsolved problems and then repeated in 2014 in monograph [9, P.330, no.16].

Problem 2. For any fixed value of $\gamma \in (0, n]$ to find the maximum of the functional

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where $n \in \mathbb{N}$, $n \geqslant 2$, $a_0 = 0$, $A_n = \{a_k\}_{k=1}^{n}$ are n-radial systems of points, such that $\mathcal{L}^{(\gamma)}\left(A_n\right) \leq 1$, $\mathcal{L}^{(0)}\left(A_n\right) \leq 1$, $\{B_k\}_{k=0}^{n}$ is any system of pairwise non-overlapping domains, such that $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{0, n}$, and to describe all extremals.

Currently it is not solved in general only partial results are known. In [10] the Problem 2 was solved for $0 < \gamma < 1$ and $n \geq 2$. In [11, 12] the authors got the solution to this problem with some restrictions on the geometry location of sets of points, namely, for $n \geq 4$ and subclass points systems satisfying condition $0 < \alpha_k \leq 2/\sqrt{\gamma}$, $k = \overline{1, n}$. In [13] the Problem 2 was solved for $\gamma \in (0, n^{0,38}]$ and $n \geq 5$. Some partial cases of the above-posed problem in the case of a unit circle $|a_k| = 1$ were considered in [14, 15, 16, 17, 18, 19].

Further let

$$I_n^0(\gamma) = r^{\gamma}\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right) \tag{1}$$

where d_k and D_k are, respectively, poles and circular domains of the quadratic differential

$$G(w)dw^2 = -\frac{(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2}dw^2.$$

Denote

$$Q_n(\gamma) = \frac{\left[2^n \frac{2}{\sqrt{\gamma}} \left(2 - \frac{2}{\sqrt{\gamma}}\right)^{n-1} (n-1)^{-(n-1)}\right]^{1-\frac{\gamma}{n}}}{\left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}}. \tag{2}$$

We obtain the following result.

Theorem 2. *Let $n \in \mathbb{N}$, $n \geq 6$, be a fixed natural number and a number γ, $\gamma \geq 1$. Then for any configuration of domains B_k and points a_k $(k = \overline{0, n})$ satisfying the conditions of Problem 2 and also provided that $\alpha_0 > \frac{2}{\sqrt{\gamma}}$, $\alpha_0 = \max\limits_{1 \leq k \leq n} \alpha_k$, the following* *sharp estimate holds*

$$\frac{I_n(\gamma)}{I_n^0(\gamma)} \leq Q_n(\gamma), \tag{3}$$

where $I_n^0(\gamma)$ and $Q_n(\gamma)$ are defined by the relations (1) and (2). If γ_n^0 be a root of the equation $Q_n(\gamma) = 1$ then for an arbitrary γ_n such that $1 \leq \gamma_n < \gamma_n^0$ the following inequality holds

$$\frac{I_n(\gamma_n)}{I_n^0(\gamma_n)} < 1.$$

Note that if we shall prove Theorem 2 we could solve the Problem 2 for $n \geq 6$, $\gamma = \gamma_n^0$, and indicate its solution for an arbitrary γ_n such that $1 < \gamma_n < \gamma_n^0$.

Proof of the Theorem 2. Let $a_0 = 0$, $A_n = \{a_k\}_{k=1}^n$ are n-radial systems of points, such that $\mathcal{L}^{(\gamma)}(A_n) \leq 1$, $\mathcal{L}^{(0)}(A_n) \leq 1$. We can assume that $0 = \arg a_1 < \arg a_2 < ... < \arg a_n < 2\pi$. Denote the number α_k, $k = \overline{1, n}$, as follows $\alpha_1 := \frac{1}{\pi}(\arg a_2 - \arg a_1)$, $\alpha_2 := \frac{1}{\pi}(\arg a_3 - \arg a_2), \ldots, \alpha_n := \frac{1}{\pi}(2\pi - \arg a_n)$. Let $\alpha_0 = \max\limits_k \alpha_k$. In the paper [12] Problem 2 was solved for an arbitrary natural number n, $n \geq 4$, $0 < \gamma \leq 0,1215n^2$ and provided that $\alpha_0 \leq \frac{2}{\sqrt{\gamma}}$. Therefore we will consider only the configurations of domains D_k and points d_k for which $\alpha_0 > \frac{2}{\sqrt{\gamma}}$.

In a similar way from theorem 5.4.1 [17] we obtain the following result

$$I_n^0(\gamma) = \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

It is easy to see that

$$I_n(\gamma) = \prod_{k=1}^n \left[r(B_0, 0) r(B_k, a_k)\right]^{\frac{\gamma}{n}} \left[\prod_{k=1}^n r(B_k, a_k)\right]^{1-\frac{\gamma}{n}}.$$

From the Lavrentev theorem [1] we obtain the following inequality

$$r(B_0, 0) r(B_k, a_k) \leq |a_k|^2.$$

Then it follows from the theorem 5.1.1 [17] that

$$\prod_{k=1}^n r(B_k, a_k) \leq 2^n \prod_{k=1}^n \alpha_k \cdot \mathcal{L}^{(0)}(A_n).$$

From the condition $\mathcal{L}^{(0)}(A_n) \leq 1$ it follows that $\prod\limits_{k=1}^n |a_k| \leq 1$. Thus

$$I_n(\gamma) \leq \left[2^n \cdot \prod_{k=1}^n \alpha_k\right]^{1-\frac{\gamma}{n}}. \tag{4}$$

Since $\sum\limits_{k=1}^{n} \alpha_k = 2$ than taking into account the Cauchy inequality between the geometric mean and arithmetic mean we have

$$\prod_{k=1}^{n} \alpha_k \le \alpha_0 \prod_{k=1,k\neq k_0}^{n} \alpha_k \le \alpha_0 \left(\frac{\sum\limits_{k=1,k\neq k_0}^{n} \alpha_k}{n-1} \right)^{n-1} = \alpha_0 \left(\frac{2-\alpha_0}{n-1} \right)^{n-1}.$$

From (4) we obtain the estimate

$$I_n(\gamma) \le \left[2^n \alpha_0 \left(\frac{2-\alpha_0}{n-1} \right)^{n-1} \right]^{1-\frac{\gamma}{n}}. \tag{5}$$

Summing the above relations and taking condition (5) into account we obtain

$$\frac{r^{\gamma}(B_0,0) \prod\limits_{k=1}^{n} r(B_k,a_k)}{I_n^0(\gamma)} \le \frac{\left[2^n \cdot \frac{2}{\sqrt{\gamma}} \left(2-\frac{2}{\sqrt{\gamma}}\right)^{n-1} (n-1)^{-(n-1)} \right]^{1-\frac{\gamma}{n}}}{\left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}} \right)^{2\sqrt{\gamma}}}.$$

Thus the inequality (3) is proved. Further we prove that the root of the equation $Q_n(\gamma) = 1$ exists for any $n \ge 6$. It is easy to see that $Q_n(1) = 0$ for every n. On the other hand

$$Q_n(n) = \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}} \right)^{2\sqrt{n}} \left(1-\frac{1}{n}\right)^{n+1} \left(\frac{n}{4}\right)^{n+1} \gg 1.$$

In this way $Q_n(1) = 0$ and $Q_n(n) > 1$. The root of the equation $Q_n(\gamma) = 1$ exists and belongs to the interval $(1, n)$. We can easily see that the function

$$\left[2^n \cdot \frac{2}{\sqrt{\gamma}} \left(2-\frac{2}{\sqrt{\gamma}}\right)^{n-1} (n-1)^{-(n-1)} \right]^{1-\frac{\gamma}{n}}$$

is monotonically increasing with respect to γ on the interval $(1, n]$. We further investigate the function $I_n^0(\gamma)$. It is elementary to verify that

$$(I_n^0(\gamma))' = I_n^0(\gamma) \left(\frac{1}{n} \ln\left(\frac{4\gamma}{n^2-\gamma}\right) + \frac{1}{\sqrt{\gamma}} \ln\left(\frac{n-\sqrt{\gamma}}{n+\sqrt{\gamma}}\right) \right).$$

In this case we shall say that the function $I_n^0(\gamma)$ decreases for fixed $n \ge 6$ and $\gamma \in (1, n]$. We agree to say that

$$I_n^0(\gamma_n^0) \le I_n^0(\gamma_n).$$

Taking last condition and property of monotonic increase of the function

$$\left[2^n \cdot \frac{2}{\sqrt{\gamma}} \left(2-\frac{2}{\sqrt{\gamma}}\right)^{n-1} (n-1)^{-(n-1)} \right]^{1-\frac{\gamma}{n}}$$

into account we obtain that the function $Q_n(\gamma)$ increases monotonically with respect to γ on the interval $[1, \gamma_n^0]$ and thus

$$Q_n(\gamma_n) < Q_n(\gamma_n^0) = 1.$$

Theorem 2 is proved. $\square$

We note that if function $Q_n(\gamma)$ is monotonic, it then follows from the obvious inequalities $\gamma_2 < \gamma_1$, $Q_n(\gamma_1) < 1$, that $Q_n(\gamma_2) < 1$.

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O PROBLEMIE EKSTREMALNEJ DEKOMPOZYCJI PŁASZCZYZNY ZESPOLONEJ

S t r e s z c z e n i e

Rozpatrujemy funkcjonał określony na układzie niezachodzących na siebie obszarów. Wynik dotyczy problemu Dubinina-poszukiwania maksimum iloczynu promieni wpisanych kół w niezachodzące na siebie obszary zawierające punkty okręgu jednostkowego i potęgę γ promienia wpisanego kola w obszar zawierający początek układu współrzędnych. Problem został sformułowany w 1994r. w pracy Dubinina, a następnie powtórzony w monografii tegoż autora z roku 2014. Problem nie jest rozwiązany w ogó;nym przypadku. W obecnej pracy problem w postaci dotyczącej bardziej ogólnego układu punktów jest uzyskany dla pewnych konkretnych wartości n oraz γ.

Słowa kluczowe: promień wewnętrzny obszaru, ekstremalna dekompozycja płaszczyzny zespolonej